Conceptual Blending as a creative meta-generator of mathematical concepts: Prime Ideals and Dedekind Domains as a Blend

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Abstract. Conceptual blending is presented as a meta-generator of mathematical concepts, by means of showing, among others, the following examples: The conceptual space of prime ideals over containment-division rings is explicitly presented, by means of an implementation in HETS, as a blend (colimit) of the conceptual space of ideals over a commutative ring with unity and the one of prime numbers over a set with a "product" operation with neutral element. Besides, a new equivalent form of being a Dedekind domain is presented allowing to express the conceptual space of prime ideals over a Dedekind domain as a blend of the space of ideals over a Noetherian ring and the one of prime numbers as before.[1]

Keywords: prime ideals, blend, colimit, Dedekind domain, containment-division ring

Introduction

In past years, conceptual blending (see Fauconnier and Turner [6]) has grown in importance in mathematical and logical domains. This cognitive process can be understood as kind of mind's natural way of combining two concepts identifying certain commonalities between them and finally, "blending" them in such a way that a new emergent meaning appears.

The following fundamental notions (blends) are examples of the growing prominence of blending: the integer, rational and real numbers; the Grothendieck group (in general, the algebraic as well as the topological K-theory) and the notions of generic points and motives in algebraic geometry, among others [1].

[1] This work should be cited as follows: Gomez-Ramirez D. Conceptual Blending as a creative meta-generator of mathematical concepts: Prime Ideals and Dedekind Domains as a Blend. C3GI at UNILOG 2015, Workshop on Computational Creativity, Concept Invention, and General Intelligence Tarek B Besold et al. (eds.) Publications of the Institute of Cognitive Sciences, PICS series, University of Osnabrück Vol 2. 2015.

A more formal approach to concept blending given in terms of colimits in the theory of categories ([8], [9] and [10]), has allowed for the reconstruction of the set of the complex numbers as a blend [12]. Furthermore, by identifying the notion of colimit with the one of pushout, one can reconstruct typical notions of modern algebra like finite dimensional vector spaces or tensor products of algebras as blends.

Moreover, the concepts of prime ideals and containment-division rings (CDR-s) can be also seen as blends [7]. In this paper, we present a complete implementation in HETS [15] of the notion of a prime ideal over a CDR as a blend of the notion of an ideal over a commutative ring with unity and the one of a prime number over a more general algebraic structure as the one of the integers with the product as roughly indicated in [7].

In [7] we present the concept of a prime ideal of a commutative ring with unity ([11] and [5]) as a sort of partial (or weaken) colimit (i.e. a colimit considering some axioms of the input theories) between the concepts of an ideal of a commutative ring with unity (enriched with the collection of all the ideals of the corresponding ring) and the concept of a prime number of the integers.

In order to obtain the desired conceptual space the authors in [7] consider a more general version of the prime numbers, namely, a monoid $(Z, *, 1)$ with an "special" divisibility relation $\lfloor$. Besides, the generic space would capture just the syntactic correspondences to be identified in the blending space. This is because, the blend, as a colimit, is essentially the union of the collections of axioms given on each space, but doing at the same time the corresponding syntactic identifications.

By slightly modifying the input conceptual spaces, we obtain, in this paper, one of the most fundamental concepts of algebraic number theory, i.e., the one of Dedekind domain [4, Theorem 37.1], (together with the collection of ideals and prime ideals) as a blend of the concepts of noetherian domains (with the set of ideals and prime ideals) and again a version of the prime numbers in a very elementary form of the integers, but adding the explicit axiom defining the upside-down divisibility relation (see the implementation in section 1). In fact, we present a new equivalent form of Dedekind domains (see §2) based on a containment-division condition, which suggests a new class of commutative rings called containment-division rings (CDR) which are shortly mentioned in [7].

It is worthwhile to mention that the concept of an ideal was discovered by Dedekind, after studying the work of Kummer on "ideal numbers" on cyclotomic fields, in order to find a more general ("ideal") entity, generalizing the notion of a number, such that the unique factorization theorem could hold over this new entities on a suitable commutative ring.

1 Implementation for prime ideals over CDR-s as a blend

Firstly, let us define the following class of commutative rings with unity.

Definition 1. *A commutative ring with unity R is a containment-division ring (CDR) if for any two ideals $I, J \subseteq$, it holds that $I \subseteq J$ if and only if J divides I as ideals, i.e., there exists an ideal D such that $I = D * *J$ (for the formal definitions of ideals and products of ideals see the corresponding axioms in the implementation below).*

On this section, we reconstruct the conceptual space of prime ideals of a CDR as a blend (colimit in HETS) of the conceptual space of ideals of a commutative ring with unity and the conceptual space of prime elements of a very general version of the integers.

It is also important to note that for this implementation we looked for a minimal set of axioms such that the semantic interpretation can be uniquely determined. It is always possible to construct an implementation with additional axioms given by properties that could be logically derived from the main axioms, (e.g. the set theoretical properties of the containment relation for subsets of a set) but these properties are secondary ones. Meanwhile, the essential ones are those that define the arithmetic of the ring, of an ideal and of the set of ideals of the ring.

```
logic CASL
%%PRIME IDEAS OVER CDR-s AS A BLEND


spec IdealsOfRing =
sort RingElt                %% sort of Ring Elements
sort SubSetOfRing           %% sort of parts of this ring
pred IsIdeal : SubSetOfRing %% when a subset is an ideal
op  0 : RingElt
op  1 : RingElt
op  __*__ : RingElt * RingElt -> RingElt
op  __+__ : RingElt * RingElt -> RingElt

pred __isIn__ : RingElt * SubSetOfRing

sort Ideal = { I : SubSetOfRing . IsIdeal(I) }
op  R : Ideal        %% the Ring as an ideal
op    __ ** __: Ideal * Ideal -> Ideal, unit R
%%Definition of the predicate of containment

pred __issubsetOf__ : Ideal * Ideal

forall A,B : Ideal
. A issubsetOf B <=> forall a: RingElt. a isIn A => a isIn B

%% axiomatization of a commutative Ring with unity
```

```
forall x : RingElt; y : RingElt . x + y = y + x
forall x : RingElt; y : RingElt; z : RingElt
. (x + y) + z = x + (y + z)
forall x : RingElt . x + 0 = x /\ 0 + x = x
forall x : RingElt . exists x' : RingElt . x' + x = 0
forall x : RingElt; y : RingElt . x * y = y * x
forall x : RingElt; y : RingElt; z : RingElt
. (x * y) * z = x * (y * z)
forall x : RingElt . x * 1 = x /\ 1 * x = x
forall x, y, z : RingElt . (x + y) * z = (x * z) + (y * z)
forall x, y, z : RingElt . z * (x + y) = (z * x) + (z * y)

%%axioms for Ideal

forall I: SubSetOfRing. IsIdeal(I) <=>
( forall a,b,c : RingElt
.( (a isIn I => a isIn R)
/\  0 isIn I)
/\ (a isIn I /\ c isIn R => (c * a) isIn I)
/\ (a isIn I /\ b isIn I /\ c isIn R
 /\ b + c = 0 => a + c isIn I ))


%% Definition of the product of ideals without subindexes

forall A,B: Ideal
. forall a,b: RingElt. (a isIn A /\ b isIn B) => a*b isIn A**B
. forall D: Ideal. (forall a,b: RingElt
. (a isIn A /\ b isIn B) => a*b isIn D)
=> A**B issubsetOf D
end

%%%axioms defining a very simple version of the integers,
%%%considered with an operation * with neutral element,
%%% a binary relation || (upside-down divisibility relation)
%%% and a primality axiom.

spec SimpleInt=
sort SimpleElem
ops 1:  SimpleElem
__ x __: SimpleElem * SimpleElem -> SimpleElem, unit 1
preds __ || __: SimpleElem * SimpleElem
IsPrime : SimpleElem
%Def_upsidedownDivisilityRelation%
forall x,y: SimpleElem
```

```
. x || y <=> (exists c: SimpleElem. x = y x c)


%% subsort of primes
sort SimplePrime = { p : SimpleElem . IsPrime(p) }

forall p:SimpleElem .
IsPrime(p) <=>
(forall a,b: SimpleElem
. a x b || p => a || p \/ b || p      %Def_primality%
/\ not (p = 1))
end

%%% Generic space

spec Gen=
sort Generic
ops S: Generic
__ gpr __: Generic * Generic -> Generic, unit S
pred gcont: Generic * Generic
end

view I1: Gen to IdealsOfRing =
Generic |-> Ideal, S |-> R,
__ gpr __ |-> __ ** __, gcont |-> __issubsetOf__

view I2: Gen to SimpleInt =
Generic |-> SimpleElem, S |-> 1,
__ gpr __ |-> __ x __, gcont |-> __ || __

spec Colimit = combine I1, I2
```

Now, seeing "RingElt" as the sort containing the elements of the ring S, one can obtain, by computing the blend in HETS (as a colimit), the (blend) theory corresponding to the axioms defining a CDR, denoted by S; the set of all its ideals, denoted by Generic; the set of all its prime ideals, denoted by SimplePrime; and a primality predicate, denoted by IsPrime:

```
logic CASL.SulFOL=

sorts Generic, RingElt, SimplePrime, SubSetOfRing
sorts SimplePrime < Generic; Generic < SubSetOfRing
op 0 : RingElt
op 1 : RingElt
op S : Generic
op __*__ : RingElt * RingElt -> RingElt
op __+__ : RingElt * RingElt -> RingElt
```

```
op __x__ : Generic * Generic -> Generic
pred IsIdeal : SubSetOfRing
pred IsPrime : Generic
pred __isIn__ : RingElt * SubSetOfRing
pred gcont : Generic * Generic

forall I : SubSetOfRing . I in Generic <=> IsIdeal(I)
%(Ax1)%

forall x : Generic . x x S = x %(ga_right_unit___**__)%

forall x : Generic . S x x = x %(ga_left_unit___**__)%

forall A, B : Generic
. gcont(A, B) <=> forall a : RingElt . a isIn A => a isIn B

%(Ax4)%

forall x, y : RingElt . x + y = y + x %(Ax5)%

forall x, y, z : RingElt . (x + y) + z = x + (y + z)%(Ax6)%

forall x : RingElt . x + 0 = x /\ 0 + x = x %(Ax7)%

forall x : RingElt . exists x' : RingElt . x' + x = 0%(Ax8)%

forall x, y : RingElt . x * y = y * x %(Ax9)%

forall x, y, z : RingElt . (x * y) * z = x * (y * z)%(Ax10)%

forall x : RingElt . x * 1 = x /\ 1 * x = x %(Ax11)%

forall x, y, z : RingElt
. (x + y) * z = (x * z) + (y * z) %(Ax12)%

forall x, y, z : RingElt
. z * (x + y) = (z * x) + (z * y) %(Ax13)%

forall I : SubSetOfRing
. IsIdeal(I)
 <=> forall a, b, c : RingElt
. ((a isIn I => a isIn S) /\ 0 isIn I)
/\ (a isIn I /\ c isIn S => c * a isIn I)
/\ (a isIn I /\ b isIn I /\
c isIn S /\ b + c = 0 => a + c isIn I)
                                                %(Ax14)%
forall a : RingElt; A : Generic
. a generates A
  <=> forall c : RingElt
. c isIn A => exists d : RingElt . c = a * d
```

```
                                                            %(Ax16)%

forall A, B : Generic; a, b : RingElt
. a isIn A /\ b isIn B => a * b isIn A x B          %(Ax17)%

forall A, B, D : Generic
. (forall a, b : RingElt
   . a isIn A /\ b isIn B => a * b isIn D)
  => gcont(A x B, D)                                %(Ax18)%

forall x, y : Generic
. gcont(x, y) <=> exists c : Generic . x = y x c  %(Ax3)%

forall p : Generic . p in SimplePrime <=> IsPrime(p)
%(Ax4_19)%

forall p : Generic
. IsPrime(p)
  <=> (forall a, b : Generic
       . gcont(a x b, p) => gcont(a, p) \/ gcont(b, p))
      /\ not p = S                                 %(Ax5_20)%
```

It is worthwhile to mention that the definition of CDR-s was obtained after computing this implementation and observing that the condition given by *(Ax3)* of the former implementation express, after computing the blending, a new non-expected condition, which is exactly the one used in the definition of a CDR. Therefore, it could be seen as a form of "creative" result coming from the blending process.

2 Prime ideals over Dedekind noetherian domains as a blend

The containment-division condition is very close related to the one defining a Dedekind domain, i.e., an integral domain such that every proper ideal can be written as a finite product of ideals [4, Theorem 37.1 and 37.8]. Effectively, if we add the property of being Noetherian [5], then both notions are equivalents:

Theorem 1. *Let R be an integral domain, i.e., a commutative ring with unity without zero divisors. Then the following two conditions are equivalent:*

1. *R is a dedekind domain.*
2. *R is a noetherian CDR.*

Proof. $1 \Rightarrow 2$.

Every Dedekind domain is Noetherian [4, Theorem 37.1]. Besides, the CDR condition is a well-known property of Dedekind domains (see for example [16, Fundamental Theorem of OAK-s]. In fact, for any ideals $I, J \in \mathrm{Id}(R)$, if $I \subseteq J$, then by the unique factorization theorem for ideals in R [4, Theorem 37.11] and by considering the localizations on the prime ideals appearing on their factorizations one sees immediately that J divides I.

$2 \Rightarrow 1$.

Let I be an proper ideal of R. If I is prime, then clearly we can express I as the product of one prime ideal. If not, let P_1 be a prime ideal of R such that $I \subseteq P_1$. Then, due to the fact that R is a CDR, a proper ideal Q_1 such that $I = Q_1 P_1$ exists. Now, if Q_1 is a prime ideal, then we can clearly express I as a product of two ideals. Otherwise, let us choose again a prime ideal P_2 containing Q_1. So, analogously there is another proper ideal Q_2 such that $Q_1 = Q_2 P_2$. If Q_2 is prime, then we can express $I = Q_2 P_2 P_1$ as a finite product of prime ideals. Otherwise, we continue inductively in the same fashion. If after finitely many steps some Q_r is prime, then we can write I as a finite product of prime ideals. In another case, we obtain an ascending chain of ideals

$$Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \cdots \subseteq Q_n \subseteq \cdots$$

Now, since R is Noetherian, this sequence is stationary (i.e., there exists some $m \in \mathbb{N}$ such that for all $i \geq m$, $Q_i = Q_m$ [2, Proposition 6.2]). Furthermore, $Q_m = Q_{m+1} P_{m+1} = Q_m P_{m+1}$ and so $Q_m = Q_m^i P_{m+1} \subseteq Q_m^i$, for all $i \in \mathbb{N}$. Therefore, I and Q_m are contained in the intersection of all the powers of Q_m, $\cap_{i \geq 1} Q_m^i$, which is the zero ideal due to Krull's Intersection Theorem [5, Corollary 5.4]. In conclusion, $I = (0)$, which is in our case a prime ideal. So, R is a Dedekind domain.

As an immediate corollary of this theorem, we see that in the setting of noetherian domains the concept of a CDR is equivalent to the one of a Dedekind domain.

Remark 1. On the other hand, if R is not a Dedekind domain, but for example a unique factorization domain (UFD), then R is not, in general, a CDR. For example, when $R = \mathbb{Z}[T]$, one can check that the ideals $X = (2)$ and $Y = (2, T)$ gives a counterexample.

It suggests that in the setting of commutative rings with unity, the class of CDR-s is an intermediate new class of rings.

Now, let us consider as our first conceptual space the space of the implementation called "IdealOfRing" with the extra condition of being a notherian domain, i.e.,

$$(\forall a, b \in R)(ab = 0 \rightarrow a = 0 \vee b = 0),$$

and the noetherian property:

Definition 2. *A commutative ring with unity R is noetherian, if for any ideal A in R, there exists $a_1, \cdots a_n \in A$ such that every element a can be written as a linear combination of these elements: there exists $b_1, \cdots, b_n \in R$ such that $a = b_1 a_! + \cdots + b_n a_n$.*

On the other hand, let us consider as second conceptual space, the space of the implementation called "SimpleInt". In particular, we include the axiom defining the upside-down divisibility relation:

$$(\forall a, b \in Z)(a\lfloor b \leftrightarrow (\exists c \in Z)(a = cb)).$$

Furthermore, let us choose the same generic space and blend morphisms as in the example of "Prime ideals as blends" presented in [7].

Then, if we do the (total) blend of the corresponding spaces, we obtain the former blend space of the implementation (commutative ring with unity, its set of ideals and prime ideals and a predicate for the prime ideals) plus the stronger condition for the ring S of being a Noetherian domain. Now, when we translate the corresponding version of the former (upside-down divisibility) axiom we obtain

$$(\forall a, b \in G)(a \subseteq b) \leftrightarrow (\exists c \in G)(a = c \cdot_\iota b),$$

where G denotes the set (sort) of ideals of R (see the sort "Generic" in the implementation).

Now, this condition means exactly being a CDR, as in the axiom 3 of the blend space in the implementation.

Therefore, by Theorem 1, we obtain as blend the conceptual space of a Dedekind domain with its collection of ideals and prime ideals and a primality predicate.

It is worthwhile to mention that the concept of an ideal was firstly discovered by Dededind, after studying the work of E. Kummer about "ideal numbers", in order to find a more general ("ideal") entity, generalizing the notion of an integer number, such that the unique factorization theorem on the integers (i.e., the fundamental theorem of arithmetic) could hold over this new "ideal" objects within a suitable number field [3] and [14]. Subsequently, Dedekind domains were defined as the canonical "suitable" rings allowing an unique factorization theorem for ideals.

Finally, one could say that the original invention of Dedekind domains had as main motivating source a metaphorical quest for extending the unique factorization properties coming from the integers.

On the other hand, we recover here again (an equivalent version of) the notion of Dedekind domain (equipped with ideals and prime ideals) just using collections of axioms defining quite more general arithmetical standard properties, not only in the case of the definition of ideals (whose properties are basically the properties defining an abelian group adding an absorption property for the product), but also a very more general concept than the one given by the monoid $(\mathbb{Z}, \cdot, 1)$ of the integers with the product operation, such that one cannot even derived from these axioms (see the concept "SimpleInt" in the former implementation) the existence of a finite factorization of elements of this structure in terms of the corresponding "prime" elements (see "SimplePrime").

So, the way of re-discovering the notion of a Dedekind domain presented here seems to offer new conceptual sources.

3 Conclusions

Mathematics had evolved as one of the fundamental structural languages use to describe the laws of our physical world. In fact, one could say that almost any modern technological device or tool was born and is based on an specific mathematical framework (e.g. computers, GPS technology, robots, etc...). It implies that the way in which we develop and discover mathematics has necessarily a cognitive component. This perspective was developed in a monist way by Lakoff and Núñes [13].

On the other hand, the presented examples suggest that (the "colimit" approach to) conceptual blending can be seen as a cognitive and universal meta-mathematical procedure, which can be seen as a "meta-generator" of mathematical definitions, which is, omnipresent among most important mathematical fields.

Finally, this contribution is aimed to give new insights in order to understand better how human-level creativity works in the specific field of pure mathematics.

Acknowledgments. The author would like to thank Alan Smaill and Felix Bou for the help in order to codifying the former implementation. Also, he wishes to thank Ulrich Von der Ohe, Terese Rutkowski and all the members of the COINVENT project.

References

1. J. Alexander, "Blending in mathematics", *Semiotica*, Volume 187, pp: 1-48, 2003.
2. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.
3. N. Bourbaki *Commutative Algebra 1-7*, Springer Verlag, 1989.
4. Eds. A. Coleman and P. Ribenboim. "Multiplicative Ideal Theory", *Quenn's Papers in Pure and Applied Mathematics*, Ontario, Canada, 1992.
5. D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, 1995.
6. G. Fauconnier and M. Turner *The Way We Think*, Basic Books, New York, 2003.
7. F. Bou, J. Corneli, D. A. J. Gomez-Ramirez, E. Maclean, A. Pease, M. Schorlemmer and A. Smaill, "The role of blending in mathematical invention", *accepted in ICCC'15*, Ohio, USA, 2015.
8. J. Gougen, "An introduction to algebraic semiotic with application to user interface design. In *Computation for methaphors, analogy and agents*, edited by C. L. Nehaniv. volume 1562, pp- 242-291. 1999.
9. J. Gougen. "Towards a design theory for virtual worlds: algebraic semiotics and scientific visualization as a case study". In *Proceedings Conference on Virtual Worlds and Simulation (Phoenix AZ, 7-11 January 2001)*, edited by C. Landauer and K. Bellman. pp: 298-303, Society for Modelling and Simulation, 2001.
10. J. Gougen. "Steps towards a design theory for virtual worlds" In *Developing future interactive systems*, edited by M. Sanchez-Segura, Idea Group publishing, pp: 116-128, 2003.

11. A. Grothendieck and J. Dieudonné. *Eléments de Géométrie Algébrique* I. Springer, 1971.
12. J. Fleuriot, E. Maclean, and D. Winterstein, “Reinventing the complex numbers”, in *Workshop at ECAE*, edited by T. Besold, A. Smaill others, volume 1 of PICS, Prague, 2014.
13. G. Lakoff and R. Núñes, *Where Mathematics Comes From*, Basic Books, 2000.
14. F. Lemmermeyer, "Jakobi and Kummer’s ideal numbers", in Arxiv: http://arxiv.org/pdf/1108.6066.pdf.
15. C. Maeder, T. Mossakowski and M. Codescu. "Hets user guide version 0.99." *http://www.informatik.uni- bremen.de/agbkb/forschung/formal methods/CoFI/hets/UserGuide.pdf*, 2014.
16. E. Weiss *Algebraic Number Theory*, McGraw-Hill, 1963.